On the Scattering of Light by Spherical Shells, and by Complete Spheres of Periodic Structure, when the Refractivity is Small.

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The problem of a small sphere of uniform optical quality has been treated in several papers.* In general, the calculations can be carried to an arithmetical conclusion only when the circumference of the sphere does not exceed a few wave-lengths. But when the relative refractivity is small enough, this restriction can be dispensed with, and a general result formulated.

In the present paper some former results are quoted, but the investigation is now by an improved method. It commences with the case of an infinitely thin spherical *shell*, from which the result for the complete uniform sphere is derived by integration. Afterwards application is made to a complete sphere, of which the structure is symmetrical but periodically variable along the radius, a problem of interest in connection with the colours, changing with the angle, often met with in the organic world.

The specific inductive capacity of the general medium being unity, that of the sphere of radius R is supposed to be K, where K − 1 is very small. Electric displacements being denoted by f, g, h, the primary wave is taken to be

$$h_0 = e^{int}\,e^{ikx}, \tag{1}$$

so that the direction of propagation is along x (negatively), and that of vibration parallel to z. The electric displacements in the scattered wave, so far as they depend upon the *first power* of (K − 1), have at a great distance the values

$$f_1, g_1, h_1 = \frac{k^2\mathrm{P}}{4\pi r}\left(\frac{\alpha\gamma}{r^2}, \frac{\beta\gamma}{r^2}, -\frac{\alpha^2+\beta^2}{r^2}\right), \tag{2}$$

in which $$\mathrm{P} = -(\mathrm{K}-1)\,.\,e^{int}\iiint e^{ik(x-r)}\,dx\,dy\,dz. \tag{3}$$

In these equations r denotes the distance between the point (α, β, γ), where the disturbance is to be estimated, and the element of volume $(dx\,dy\,dz)$ of the obstacle. The centre of the sphere R will be taken as the origin of co-ordinates. It is evident that, so far as the secondary ray is concerned, P

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depends only on the angle (χ) which this ray makes with the primary ray. We will suppose that $\chi = 0$ in the direction backwards along the primary ray, and that $\chi = \pi$ along the primary ray continued. The integral in (3) may then be found in the form

$$\frac{2\pi R^2 e^{-ik\rho}}{k \cos \frac{1}{2}\chi} \int_0^{\frac{1}{2}\pi} J_1(2kR \cos \tfrac{1}{2}\chi \,.\, \cos \phi) \cos^2 \phi \, d\phi, \tag{4}*$$

ρ denoting the distance of the point of observation from the *centre* of the sphere. In the paper of 1914 I showed that the integral in (4) can be simply expressed by circular functions in virtue of a theorem given by Hobson, so that

$$P = -(K-1) \,.\, 4\pi R^3 \,.\, e^{i(nt-k\rho)} \left(\frac{\sin m}{m^3} - \frac{\cos m}{m^2} \right), \tag{5}$$

where
$$m = 2kR \cos \tfrac{1}{2}\chi. \tag{6}$$

In (5) the optical quality of the sphere, expressed by (K−1), is supposed to be uniform throughout. In view of an application presently to be considered, it was desired to obtain the expression for a spherical *shell* of infinitesimal thickness dR, from which could be derived the value of P for a complete symmetrical sphere whose optical quality varies along the radius. The required result is obtained at once from (5) and (6) by differentiation. We find

$$dP = -(K-1) \,.\, 4\pi R^2 dR \,.\, e^{i(nt-k\rho)} \,.\, \sin m/m, \tag{7}$$

expressing the value of P for a spherical shell of volume $4\pi R^2 dR$. The simplicity of (7) suggested that the reasoning by which it had been arrived at is needlessly indirect, and that a better procedure would be an inverse one, in which (7) was established first, and the result for the complete sphere derived from it by integration. And this anticipation was easily confirmed.

Commencing then with a spherical shell of centre O and radius OA equal to R, let xO be the direction of the primary and Oρ that of the secondary ray (fig. 1). Draw Oζ in the plane of Ox, Oρ, and bisecting the angle between these lines and let ζ be a co-ordinate measured from O in the direction Oζ, so that the plane AOA, perpendicular to Oζ, is represented by $\zeta = 0$. The angle xOζ is $\frac{1}{2}\chi$, as in our former notation. We have now to consider the phases represented by the factor $e^{ik(x-r)}$ in P. For the point O, $x = 0$, $r = \rho$, and the exponential factor is $e^{-ik\rho}$. As in the ordinary theory of specular reflection, the same is true for every point in the plane AOA and therefore for the element of surface at AA whose volume is $2\pi R dR d\zeta$. For points in a plane

* Given in the 1881 paper.

BB parallel to AA at a distance ζ the linear retardation is $-2\zeta\cos\frac{1}{2}\chi$, as in the theory of thin plates; and the exponential factor is $e^{-ik\rho}\,e^{2ik\zeta\cos\frac{1}{2}\chi}$. The

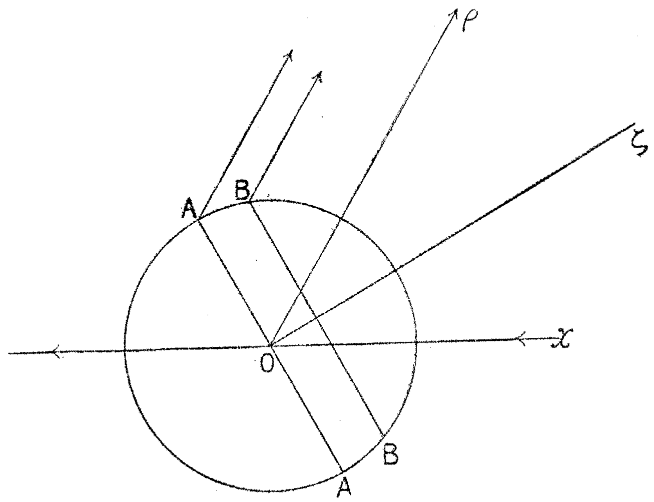

FIG. 1.

elementary volume at BB is still expressed by $2\pi R\,dR\,d\zeta$, and accordingly by (3)

$$dP = -(K-1)\,.\,2\pi R\,dR\,.\,e^{i(nt-k\rho)}\int_{-R}^{+R} d\zeta\, e^{2ik\zeta\cos\frac{1}{2}\chi}. \tag{8}$$

The integral in (8) is $2R\sin m/m$, m being given by (6), and we recover (7) as expressing the value of dP for a spherical shell of volume $4\pi R^2\,dR$.

The value of dP for a spherical shell having been now obtained independently, we can pass at once by integration to the corresponding expression for a complete sphere of uniform optical quality, thus recovering (5) by a simpler method not involving Bessel's functions at all. And a comparison of the two processes affords a demonstration of Hobson's theorem formerly employed as a stepping stone.

When P is known, the secondary vibration is given by (2), in which we may replace r by ρ. So far as it depends upon P, the angular distribution, being a function of χ, is symmetrical round Ox, the direction of primary propagation. So far as it depends on the other factors $\alpha\gamma/\rho^2$, etc., it is the same as for an infinitely small sphere; in particular no ray is emitted in the direction defined by $\alpha = \beta = 0$, that is in the direction of primary vibration. There is no limitation upon the value of R if $(K-1)$ be small *enough*; but the reservation is important, since it is necessary that at every point of the obstacle the retardation of the primary waves *due to the obstacle* be negligible.

When R is great compared with $\lambda\,(=2\pi/k)$, m usually varies rapidly with R or k, and so does P, as given for the complete uniform sphere in (5). An exception occurs when χ is nearly equal to π, that is when the secondary ray is nearly in the direction of the primary ray continued ($\beta = \gamma = 0$). In this case m is very small,

$$\frac{\sin m}{m^3} - \frac{\cos m}{m^2} = \frac{1}{3},$$

and $|P|$ is independent of k, and is proportional to R^3. The *intensity* is then

$$|P^2| = \frac{16(K-1)^2 \pi^2 R^6}{9}. \tag{9}$$

The haze immediately surrounding a small source of light seen through a foggy medium is of relatively great intensity. And the cause is simply that the contributions from the various parts of a small obstacle agree in phase.

But in general when R is great, so also is m, and $|P|$ varies rapidly and periodically with k along the spectrum. We might then be concerned mainly with the mean value of $|P^2|$. Now

$$|P^2| = (K-1)^2 \cdot 4^2 \pi^2 R^6 (\sin m - m \cos m)^2 m^{-6},$$

of which the mean value is

$$(K-1)^2 \cdot 8\pi^2 R^6 (1 + m^2) m^{-6},$$

or approximately, since m is great,

$$(K-1)^2 \cdot 8\pi^2 R^6 m^{-4}.$$

When we introduce the value of m from (6), this becomes

$$\text{Mean } |P^2| = \frac{(K-1)^2 \pi^2 R^2}{2k^4 \cos^4 \frac{1}{2}\chi} = \frac{(K-1)^2 R^2 \lambda^4}{32\pi^2 \cos^4 \frac{1}{2}\chi}. \tag{10}$$

The occurrence of λ^4 shows that this is in general very small in comparison with (9).

If, instead of a sphere of uniform quality, we have to deal with one where $(K-1)$ is variable, we must employ (7). The case of greatest interest is when $(K-1)$, besides a constant, includes also a periodic part. For the constant part the integration proceeds as before, and for the periodic part, where $(K-1)$ varies as a circular function of R, it presents no difficulty. It may suffice to consider the particular case where $(K-1)$ is proportional to $\sin m$, m as before being given by (6); for this supposition evidently leads to a large augmentation of P, analogous to what occurs in crystals of chlorate of potash, to which a plane periodic structure is attributed.* It will be

* 'Phil. Mag.,' vol. 26, p. 256 (1888); 'Scientific Papers,' vol. 3, p. 204.

observed that the wave-length of the structure now supposed varies with χ, as well as with k or λ. Thus, if $K-1 = \beta\, m$,

$$P = -\pi\beta\, e^{i(nt-kp)} R^3 \frac{m^2 - m \sin 2m + \frac{1}{2}(1-\cos 2m)}{m^3}, \tag{11}$$

when the integration is taken for a complete sphere of radius R. If m is moderately great, that is, if R be a large multiple of λ, the first term on the right of (11) preponderates, and we may use approximately

$$P = -\frac{\pi\beta R^2 e^{i(nt-kr)}}{2k \cos\frac{1}{2}\chi}. \tag{12}$$

Thus, if $(K-1)$ has no constant part,

$$|P| = \frac{\pi\beta R^2}{2k\cos\frac{1}{2}\chi} = \frac{\beta R^2\lambda}{4\cos\frac{1}{2}\chi}. \tag{13}$$

The relation between the wave-length of the structure (Λ) and that of the light is expressed by

$$\Lambda = \tfrac{1}{2}\lambda/\cos\tfrac{1}{2}\chi. \tag{14}$$

It seems probable that a structure of this sort is the cause of the remarkable colours, variable with the angle of observation, which are so frequent in beetles, butterflies, and feathers.